

ALGEBRAIC VARIETIES WITH SEMIALGEBRAIC UNIVERSAL COVER

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The study of algebraic varieties using their universal cover goes back to Euler (for elliptic curves), Abel (for Abelian varieties) and Poincaré (for curves of genus ≥ 2). A general approach was initiated by Shafarevich [Sha74, Sec.IX.4]; see [Kol95, Kat97, BK98, EKPR09] and the references there for later results and surveys.

In the classical examples, the universal cover is rather simple (\mathbb{C}^n or a bounded symmetric domain), which makes it possible to get detailed information about a variety using its universal cover. This leads to our basic question:

Which smooth projective varieties have “simple” universal cover?

There are at least three ways to define what “simple” should mean, depending on which properties of the universal cover \tilde{X} we focus on.

- (Topology) \tilde{X} is homotopic to a finite CW complex.
- (Complex analysis) \tilde{X} is a bounded domain in a Stein manifold.
- (Algebraic geometry) \tilde{X} is an algebraic variety.

The algebro-geometric variant was investigated in [CHK11] whose main result roughly says that all such examples are obtained from Abelian varieties.

Theorem 1. [CHK11, Thm.1] *Let X be a normal, projective variety over \mathbb{C} with universal cover \tilde{X} . The following are equivalent.*

- (1) \tilde{X} is biholomorphic to a quasi-projective variety.
- (2) \tilde{X} is biholomorphic to a product $\mathbb{C}^m \times F$ where $m \geq 0$ and F is a projective, simply connected variety.

(The proof in [CHK11] assumes the validity of the Abundance Conjecture [Rei83], so the theorem is unconditionally established only if $\dim X \leq 3$.)

Our aim is to extend these results to a larger class of examples, that includes the compact quotients of bounded, symmetric domains. Let $\mathbb{D} \subset \mathbb{C}^n$ be a bounded, symmetric domain in any of its usual representations. Then \mathbb{D} is not quasi-projective, but it is *semialgebraic*. That is, if we set $z_j = x_j + iy_j$, then \mathbb{D} can be defined by polynomial inequalities of the form $f(x_1, \dots, x_n, y_1, \dots, y_n) > 0$; see (7) and (17) for details. This leads to the following.

Conjecture 2. *Let X be a normal, projective variety over \mathbb{C} with universal cover \tilde{X} . The following are equivalent.*

- (1) \tilde{X} is biholomorphic to a semialgebraic open subset of a projective variety.
- (2) \tilde{X} is biholomorphic to a product $\mathbb{D} \times \mathbb{C}^m \times F$ where \mathbb{D} is a bounded symmetric domain, $m \geq 0$ and F is a normal, projective, simply connected variety.

The first part of the proof of [CHK11] shows that if (1.1) holds then $\pi_1(X)$ has a finite index abelian subgroup. The second part then shows that Theorem 1 holds if $\pi_1(X)$ is abelian.

Our main result is a non-abelian version of this second step. In effect we prove Conjecture 2 in case $\pi_1(X)$ acts properly discontinuously and freely with compact quotient on an irreducible, bounded, symmetric domain \mathbb{D} . A three step approach to the proof of Conjecture 2 is outlined in Section 3. Our results establish Step 3 and most of Step 2. However, we say nothing about Step 1 which seems to be the hardest. On the other hand, we also describe certain intermediate covers in a precise way.

Theorem 3. *Let Γ be a group acting properly discontinuously and freely with compact quotient on an irreducible, bounded, symmetric domain \mathbb{D} of dimension ≥ 2 . Let X be a smooth projective variety, $\pi_1(X) \rightarrow \Gamma$ a quotient and $\tilde{X}_\Gamma \rightarrow X$ the corresponding Γ -cover. Then the following are equivalent.*

- (1) \tilde{X}_Γ is biholomorphic to a semialgebraic subset of a projective variety.
- (2) X is a fiber bundle over \mathbb{D}/Γ .
- (3) \tilde{X}_Γ is biholomorphic to a product $\mathbb{D} \times F$ where F is a projective variety.

Note that in (3.1) we do not assume that the Γ -action is semialgebraic on \tilde{X}_Γ . As noted in [CHK11, 3.1], this is usually not true, but it holds if \tilde{X}_Γ is biholomorphic to a bounded semialgebraic domain in \mathbb{C}^n [Zai95].

Remark 4. Small changes are needed in the statement of Theorem 3 if \mathbb{D} is the unit disc in \mathbb{C} . By [Siu87, Cat91], in this case there is a morphism with connected fibers $g : X \rightarrow C$ to a smooth curve C of genus ≥ 2 , but $\pi_1(C)$ may be larger than Γ . The rest of the proof works if we replace \tilde{X}_Γ by the cover corresponding to $\pi_1(C)$.

Remark 5. Every biholomorphism of $\mathbb{D} \times \mathbb{C}^m \times F$ preserves the projections

$$\mathbb{D} \times \mathbb{C}^m \times F \rightarrow \mathbb{D} \times \mathbb{C}^m \rightarrow \mathbb{D}.$$

Thus, if the universal cover of X is $\mathbb{D} \times \mathbb{C}^m \times F$, then $\pi_1(X)$ acts on $\mathbb{D} \times \mathbb{C}^m$ and on \mathbb{D} properly discontinuously but not necessarily freely. By passing to a finite cover $X' \rightarrow X$, we may assume that the $\pi_1(X')$ -action on \mathbb{D} is also free, modulo its kernel (cf. [Sel60, p.154]). Thus if we quotient by $\pi_1(X')$, we obtain morphisms of algebraic varieties $X' \rightarrow Y' \rightarrow Z'$ such that

- (1) $X' \rightarrow Y'$ is a fiber bundle whose fiber is a normal, projective variety,
- (2) $Y' \rightarrow Z'$ is a smooth morphism whose fibers are Abelian varieties (but it need not be a holomorphic fiber bundle) and
- (3) $Z' = \mathbb{D}/\Gamma'$ for some group Γ' acting properly discontinuously and freely.

6 (Main steps of the proof). A quotient $\sigma : \pi_1(X) \rightarrow \Gamma$ is equivalent to a homotopy class of continuous maps $X \rightarrow \mathbb{D}/\Gamma$.

First, we use the purely topological results of Section 1 to prove that every map in this homotopy class is surjective.

Second, a combination of [ES64] and [Siu82] implies that there is a holomorphic map $g_\sigma : X \rightarrow \mathbb{D}/\Gamma$ that induces σ on the fundamental groups; see (7) and (8) for details. These two steps so far used only that \tilde{X}_Γ is homotopic to a finite CW complex.

The third step, which is modeled on [CHK11, Sec.2], is best explained by the special case when $Y := \mathbb{D}/\Gamma$ is a curve, X is a surface and $g : X \rightarrow Y$ has connected fibers.

Fix a normal, projective surface \bar{X} that contains \tilde{X} as an open semialgebraic subset. Let $\tilde{g} : \tilde{X} \rightarrow \tilde{Y}$ be the lifting of g and \tilde{F} a general fiber of \tilde{g} . Note that \tilde{F} has self-intersection 0 and it moves, thus it determines a morphism $\bar{g} : \bar{X} \rightarrow \bar{Y}$.

The following two observations almost contradict each other.

- (i) $\bar{g} : \bar{X} \rightarrow \bar{Y}$ is an algebraic family of curves, hence it gives an algebraic morphism of \bar{Y} to the moduli space of curves. The fibers of this moduli map are algebraic subsets of \bar{Y} .
- (ii) Each fiber of $g : X \rightarrow Y$ gives rise to $|\Gamma|$ copies of itself among the fibers of \tilde{g} . Thus the moduli map has infinite fibers.

The only way out is if the moduli map is constant and all the smooth fibers of $g : X \rightarrow Y$ are isomorphic to each other. Note that any singular fiber of g would lead to infinitely many singular fibers of \tilde{g} and of \bar{g} ; this is again impossible. Thus there are no singular fibers and $g : X \rightarrow Y$ is a fiber bundle.

7 (Bounded symmetric domains). For general reference, see [Hel62, Chap.VIII].

Let \mathbb{D} be a bounded symmetric domain and Γ a group acting properly discontinuously and freely with compact quotient \mathbb{D}/Γ .

Let X be a smooth projective variety. A group homomorphism $\sigma : \pi_1(X) \rightarrow \Gamma$ is equivalent to a homotopy class of continuous maps $X \rightarrow \mathbb{D}/\Gamma$. By Eells-Sampson [ES64], every homotopy class contains a harmonic map $g_\sigma : X \rightarrow \mathbb{D}/\Gamma$. A theorem of Siu [Siu82, Thm.6.7] says that if \mathbb{D} is irreducible and the rank of dg_σ is large enough at some point, then g_σ is either holomorphic or conjugate holomorphic. The result applies whenever g_σ is surjective and $\dim \mathbb{D} \geq 2$, which is assured by the purely topological Theorem 14. Thus, for a suitable choice of the Γ -action on \mathbb{D} we may assume that g_σ is holomorphic.

We can summarize the above considerations as follows.

Lemma 8. *Let Γ be a group acting properly discontinuously and freely with compact quotient \mathbb{D}/Γ on an irreducible, bounded, symmetric domain \mathbb{D} of dimension ≥ 2 . Let X be a smooth projective variety, $\sigma : \pi_1(X) \rightarrow \Gamma$ a quotient and $\tilde{X}_\Gamma \rightarrow X$ the corresponding Γ -cover. Assume that \tilde{X}_Γ is homotopic to a finite CW complex.*

Then there is a holomorphic map $g_\sigma : X \rightarrow \mathbb{D}/\Gamma$ that induces σ on the fundamental groups. \square

1. MAPS TO COMPACT $K(\pi, 1)$ SPACES

In this section we consider the following.

Question 9. Let X be a finite CW complex which is a $K(\pi, 1)$ and A a compact metric space mapping to X . Let \tilde{X} be the universal cover of X and \tilde{A} the corresponding cover of A (that is, the fibered product, or pull-back). We have a commutative fiber product diagram:

$$\begin{array}{ccc} \tilde{A} & \rightarrow & A \\ \tilde{\omega} \downarrow & & \downarrow \omega \\ \tilde{X} & \rightarrow & X. \end{array}$$

Suppose that \tilde{A} has some finiteness properties. What can one conclude about $A \rightarrow X$?

We get quite strong results if \tilde{A} is homotopy equivalent to a compact metric space (Theorems 14, 15, 16), but we are not sure what the optimal conclusions should

be. Most of the results in this section apply if \tilde{X} is contractible and $X = \tilde{X}/\Gamma$ for a properly discontinuous cocompact action of Γ (that is, we do not need the action to be free).

Definition 10. Suppose X is a metric space with the property that closed metric balls are compact. Let $\pi \subset \text{Isom}(X)$ be a group whose action is cocompact and properly discontinuous. Denote its action by $T_\alpha : X \rightarrow X$, so $T_\alpha T_\beta = T_{\alpha\beta}$. We call such a pair (X, π) a *proper π -space*. If we additionally fix a set $\{p_\alpha\}_{\alpha \in \pi}$ so $T_\alpha p_\beta = p_{\alpha\beta}$, then we call such a triple $(X, \pi, \{p_\alpha\}_{\alpha \in \pi})$ a *pointed proper π -space*.

For any pointed proper π -space $(X, \pi, \{p_\alpha\}_{\alpha \in \pi})$, we may rescale the metric so that the closed unit balls of radius 1 centered at the $\{p_\alpha\}_{\alpha \in \pi}$ cover all of X . All of the results in this section are indifferent to such rescalings. Therefore we shall often assume implicitly that the particular pointed proper π -space in question has this property.

Lemma 11. Let $(X, \pi, \{p_\alpha\}_{\alpha \in \pi})$ be a pointed proper π -space. Let $F^t : X \rightarrow X$ for $t \in [0, 1]$ be a homotopy between $F^0 = \text{id}_X$ and $F^1 : X \rightarrow K \subseteq X$ where K is compact.

We denote by $\pi!$ the set of total orders on π . For any finite subset $P \subset \pi$, consider the continuous function $W_F : [0, 1]^P \times \pi! \times X \rightarrow X$ defined by:

$$W_F(\{t_\alpha\}_{\alpha \in P}, \sigma, x) := \left(\prod_{\alpha \in P, \text{ordered by } \sigma} T_\alpha \circ F^{t_\alpha} \circ T_\alpha^{-1} \right)(x) \quad (11.1)$$

For $P_1 \subseteq P_2$, map $[0, 1]^{P_1} \rightarrow [0, 1]^{P_2}$ by extending by zero. Since F^0 is the identity, the functions $\{W_F\}_{P \subset \pi}$ are compatible with the directed system $\{[0, 1]^P \times \pi! \times X\}_{P \subset \pi}$, and so we really have a continuous function defined on the direct (inductive) limit, which we also denote by W_F .

We have that for every $N < \infty$ there exists $M < \infty$ such that:

- (2) $\sup_{t_\alpha \neq 0} d(p_\alpha, x) < N \implies d(W_F(\{t_\alpha\}_{\alpha \in \pi}, \sigma, x), x) < M$ and
- (3) $t_\gamma = 1$ and $\sup_{t_\alpha \neq 0} d(p_\alpha, p_\gamma) < N \implies d(W_F(\{t_\alpha\}_{\alpha \in \pi}, \sigma, x), p_\gamma) < M$.

Proof. For an order $\sigma \in \pi!$ and an element $\gamma \in \pi$, define $\sigma^\gamma \in \pi!$ by the property:

$$\beta_1 \prec_{\sigma^\gamma} \beta_2 \iff \gamma\beta_1 \prec_\sigma \gamma\beta_2$$

Then we have the following identity:

$$W_F(\{t_\alpha\}_{\alpha \in \pi}, \sigma, x) = T_\gamma W_F(\{t_{\gamma\alpha}\}_{\alpha \in \pi}, \sigma^\gamma, T_\gamma^{-1}x). \quad (11.4)$$

To prove (11.2), note that given $x \in X$, we can find $\gamma \in \pi$ such that $d(p_\gamma, x) \leq 1$. Then by (11.4) we have:

$$d(W_F(\{t_\alpha\}_{\alpha \in \pi}, \sigma, x), x) = d(W_F(\{t_{\gamma\alpha}\}_{\alpha \in \pi}, \sigma^\gamma, T_\gamma^{-1}x), T_\gamma^{-1}x)$$

Given $N < \infty$, let $P_N = \{\alpha \in \pi : d(p_\alpha, p_1) < N\}$ (this set is finite). Then the above quantity is certainly in the image of the map $[0, 1]^{P_N} \times P_N! \times B(p_1, 1) \rightarrow \mathbb{R}_{\geq 0}$ which sends $(\{t_\alpha\}_{\alpha \in P_N}, \sigma, y)$ to $d(W_F(\{t_\alpha\}_{\alpha \in P_N}, \sigma, y), y)$. The domain of this function is compact, so its range is bounded above by some $M < \infty$.

To prove (11.3), we observe that by (11.4) we have:

$$d(W_F(\{t_\alpha\}_{\alpha \in \pi}, \sigma, x), p_\gamma) = d(W_F(\{t_{\gamma\alpha}\}_{\alpha \in \pi}, \sigma^\gamma, T_\gamma^{-1}x), p_1)$$

Note that since $t_\gamma = 1$, the function corresponding to $1 \in \pi$ in the evaluation of W_F on the right hand side has image contained in K . Thus the right hand side is

in the image of the map $[0, 1]^{P_N} \times P_N! \times K \rightarrow \mathbb{R}_{\geq 0}$ which sends $(\{t_\alpha\}_{\alpha \in P_N}, \sigma, y)$ to $d(W_F(\{t_\alpha\}_{\alpha \in P_N}, \sigma, y), p_1)$. The domain of this function is compact, so its range is bounded above by some $M < \infty$. \square

The notation W_F will be used throughout this section to denote the function given in (11.1). Also, the order $\sigma \in \pi!$ will be irrelevant from now on. Thus we assume implicitly that such an order is fixed, and we suppress σ from the notation. Let us now fix a continuous (cutoff) function $w : \mathbb{R}_{\geq 0} \rightarrow [0, 1]$ with $w(t) = 1$ for $t \in [0, 1]$ and $w(t) = 0$ for $t \geq 2$.

Lemma 12. *Let (X, π) be a proper π -space, and suppose that X is contractible. Then there exists a continuous function $R : X \times X \times [0, 1] \rightarrow X$ such that $R(x, y, 0) = x$, $R(x, y, 1) = y$, and for every $N < \infty$ there exists $M < \infty$ such that:*

$$d(x, y) < N \implies d(x, R(x, y, t)) < M \quad (12.1)$$

Proof. Fix $\{p_\alpha\}_{\alpha \in \pi}$ such that $(X, \pi, \{p_\alpha\}_{\alpha \in \pi})$ is a pointed proper π -space.

Let $G^t : X \rightarrow X$ ($t \in [0, 1]$) be a contraction to p_1 , that is $G^0(x) = x$ and $G^1(x) = p_1$. We define:

$$R(x, y, t) := \begin{cases} W_G(\{2t \cdot w(d(x, p_\alpha))\}, x) & t \in [0, \frac{1}{2}] \\ W_G(\{2(1-t) \cdot w(d(x, p_\alpha))\}, y) & t \in [\frac{1}{2}, 1] \end{cases}$$

For $t = \frac{1}{2}$, the two definitions agree, since one of the functions in the composition defining $W_G(\{w(d(x, p_\alpha))\}, \cdot)$ is constant (since for all x , one of the values $w(d(x, p_\alpha))$ is equal to 1). Clearly $R(x, y, 0) = x$ and $R(x, y, 1) = y$. By (11.2), we have the desired property (12.1). \square

Proposition 13. *Notation and assumptions as in (9). Suppose that we have a homotopy equivalence $\rho : \tilde{A} \rightarrow Z$ for some compact metric space Z . Then the product map $(\rho, \tilde{\omega}) : \tilde{A} \rightarrow Z \times \tilde{X}$ is a proper homotopy equivalence.*

Proof. Fix a metric on \tilde{X} and points $\{p_\alpha\}_{\alpha \in \pi_1(X)}$ so that $(\tilde{X}, \pi_1(X), \{p_\alpha\}_{\alpha \in \pi_1(X)})$ is a pointed proper $\pi_1(X)$ -space. Metrize \tilde{A} by $d_{\tilde{A}} = d_A + d_{\tilde{X}}$; then \tilde{A} is also a proper $\pi_1(X)$ -space, and the quotient metric induces the same topology as d_A .

Denote by $\rho' : Z \rightarrow \tilde{A}$ the function paired with ρ forming the homotopy equivalence. Let $F^t : \tilde{A} \rightarrow \tilde{A}$ ($t \in [0, 1]$) be a homotopy between the identity map and $\rho' \circ \rho$. Thus for all $a \in \tilde{A}$, we have $F^0(a) = a$ and $F^1(a) \in K$ for some compact set $K = \text{im } \rho'$.

Define a map $\phi : Z \times \tilde{X} \rightarrow \tilde{A}$ as follows:

$$\phi(z, x) := W_F(\{w(d(x, p_\alpha))\}, \rho'(z))$$

By (11.3), $\phi : Z \times \tilde{X} \rightarrow \tilde{A}$ satisfies $\sup_{(z, x) \in Z \times \tilde{X}} d(x, \tilde{\omega}(\phi(z, x))) < \infty$. Thus ϕ is proper. Since A is compact, the map $(\rho, \tilde{\omega}) : \tilde{A} \rightarrow Z \times \tilde{X}$ is proper as well. We now prove both compositions are proper homotopic to the identity.

Consider first the function $(\rho, \tilde{\omega}) \circ \phi : Z \times \tilde{X} \rightarrow Z \times \tilde{X}$, which is given by:

$$(\rho(W_F(\{w(d(x, p_\alpha))\}, \rho'(z))), \tilde{\omega}(\phi(z, x))) \quad (13.1)$$

Using (12), the above function is proper homotopic via:

$$(\rho(W_F(\{w(d(x, p_\alpha))\}, \rho'(z))), R(\tilde{\omega}(\phi(z, x)), x, t))$$

to:

$$(\rho(W_F(\{w(d(x, p_\alpha))\}, \rho'(z))), x) \quad (13.2)$$

Now since Z is compact, the family $(\rho(W_F(\{t \cdot w(d(x, p_\alpha))\}, \rho'(z))), x)$ is a proper homotopy between (13.2) and $(\rho(\rho'(z)), x)$, which by definition is proper homotopic to (z, x) , i.e. the identity map.

Now consider the function $\phi \circ (\rho, \tilde{\omega}) : \tilde{A} \rightarrow \tilde{A}$, which is given by:

$$W_F(\{w(d(\tilde{\omega}(a), p_\alpha))\}, \rho'(\rho(a))) \quad (13.3)$$

We know $\rho'(\rho(a))$ is homotopic to the identity map. Thus (13.3) is homotopic to:

$$W_F(\{w(d(\tilde{\omega}(a), p_\alpha))\}, a) \quad (13.4)$$

and the homotopy is proper by (11.3). Now $W_F(\{t \cdot w(d(\tilde{\omega}(a), p_\alpha))\}, a)$ gives a homotopy between (13.4) and the identity map; this homotopy is proper by (11.2). \square

We shall have occasion below to use various flavors of (usually singular) homology and cohomology. We let H_* and H^* denote standard homology and cohomology; these are functorial with respect to homotopy classes of continuous maps. We let H_*^{lf} and H_c^* denote locally finite homology and compactly supported cohomology respectively; these are functorial with respect to proper homotopy classes of proper maps (see [Geo08, III.10–12] for some definitions and basic properties). We use \tilde{H}_* to denote reduced homology. Our coefficient group is always \mathbb{Z} .

Theorem 14. *Notation and assumptions as in (9). Assume further that either X is a manifold or that X is a (possibly singular) complex analytic space.*

If \tilde{A} is homotopy equivalent to a nonempty compact metric space, then $A \rightarrow X$ is surjective.

Proof. Let us call the homotopy equivalence $\rho : \tilde{A} \rightarrow Z$. Pick any point $x \in \tilde{X}$ where \tilde{X} is locally a manifold of dimension n , and suppose that x is not in the image of \tilde{A} . Then we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\rho \times \tilde{\omega}} & Z \times \tilde{X} \\ \tilde{\omega} \downarrow & & \downarrow p_{\tilde{X}} \\ \tilde{X} - \{x\} & \rightarrow & \tilde{X} \end{array}$$

By (13), the map $\tilde{A} \rightarrow Z \times \tilde{X}$ is a proper homotopy equivalence, and thus induces an isomorphism on H_*^{lf} . Clearly the map $Z \times \tilde{X} \rightarrow \tilde{X}$ is surjective on H_*^{lf} . Thus the map $\tilde{A} \rightarrow \tilde{X}$ is surjective on H_*^{lf} . On the other hand, certainly $H_n^{\text{lf}}(\tilde{X} - \{x\}) \rightarrow H_n^{\text{lf}}(\tilde{X})$ is not surjective: in the case X is a manifold, the fundamental class $[\tilde{X}] \in H_n^{\text{lf}}(\tilde{X})$ is not in the image, and in the case X is a complex analytic space, the fundamental class of the irreducible component of \tilde{X} containing x is not in the image. This is a contradiction, so we conclude that x is in the image of $\tilde{A} \rightarrow \tilde{X}$.

Thus every point $x \in X$ where X is locally a manifold is in the image of $A \rightarrow X$. If X is a manifold, we are done. If X is a complex analytic space, observe that the set of points where X is a manifold is dense. On the other hand, A is compact, so the image of $A \rightarrow X$ is closed. Thus we are done in this case as well. \square

Theorem 15. *Notation and assumptions as in (9). Assume further that $A \rightarrow X$ is a complex analytic map of compact complex analytic spaces and that X is normal. Suppose that \tilde{A} is homotopy equivalent to a finite CW complex.*

Then for every fiber \tilde{A}_x of \tilde{A} , there is a natural map $H_(\tilde{A}) \rightarrow H_*(\tilde{A}_x)$ whose composition with $H_*(\tilde{A}_x) \rightarrow H_*(\tilde{A})$ is the identity.*

Proof. Let U be the set of points $x \in \tilde{X}$ where \tilde{X} is locally a C^∞ -manifold over which $\tilde{A} \rightarrow \tilde{X}$ is a C^∞ -bundle. Then $\tilde{X} \setminus U$ is a closed subvariety. Suppose we prove the theorem for all $x \in U$; then the general case is done as follows. Given any fiber \tilde{A}_x , we can find an open neighborhood which deformation retracts onto it. Such an open neighborhood contains \tilde{A}_u for all $u \in U \cap N_\epsilon(x)$. Since \tilde{X} is normal, U is locally connected near u . Thus we get a well defined map $H_*(\tilde{A}) \rightarrow H_*(\tilde{A}_u) \rightarrow H_*(\tilde{A}_x)$ with the desired property. Now let us prove the result for $x \in U$.

Let $\rho : \tilde{A} \rightarrow Z$ be the homotopy equivalence from \tilde{A} to a finite CW complex Z , and let $\phi : Z \times \tilde{X} \rightarrow \tilde{A}$ be the resulting proper homotopy equivalence from (13). Let $2n$ be the dimension of \tilde{X} as a real manifold, and define $H_*(\tilde{A}) \rightarrow H_*(\tilde{A}_x)$ as follows:

$$H_*(\tilde{A}) \xrightarrow{\rho_*} H_*(Z) \xrightarrow{\times[\tilde{X}]} H_{*+2n}^{\text{lf}}(Z \times \tilde{X}) \xrightarrow{\phi_*} H_{*+2n}^{\text{lf}}(\tilde{A}) \rightarrow H_{*+2n}(\tilde{A}, \tilde{A} \setminus \tilde{A}_x) = H_*(\tilde{A}_x)$$

By construction, the maps for different nearby values of x are compatible. Now let us prove that the composition $H_*(\tilde{A}) \rightarrow H_*(\tilde{A}_x) \rightarrow H_*(\tilde{A})$ is the identity map.

Let us first consider the last few maps in this long composition, namely:

$$H_{*+2n}^{\text{lf}}(\tilde{A}) \rightarrow H_{*+2n}(\tilde{A}, \tilde{A} \setminus \tilde{A}_x) = H_*(\tilde{A}_x) \rightarrow H_*(\tilde{A})$$

This map is just given by the cap product with $\tilde{\omega}^*([\tilde{X}]) \in H_c^{2n}(\tilde{A})$, where we denote by $[\tilde{X}] \in H_c^{2n}(\tilde{X})$ the unique class whose evaluation on the fundamental class of \tilde{X} is 1. Thus we conclude that the composition $H_*(\tilde{A}) \rightarrow H_*(\tilde{A}_x) \rightarrow H_*(\tilde{A})$ is:

$$H_*(\tilde{A}) \xrightarrow{\rho_*} H_*(Z) \xrightarrow{\times[\tilde{X}]} H_{*+2n}^{\text{lf}}(Z \times \tilde{X}) \xrightarrow{\phi_*} H_{*+2n}^{\text{lf}}(\tilde{A}) \xrightarrow{\cap \tilde{\omega}^*([\tilde{X}])} H_*(\tilde{A})$$

Note that $\alpha \in H_*(Z)$ is sent to:

$$\begin{aligned} \phi_*(\alpha \times [\tilde{X}]) \cap \tilde{\omega}^*([\tilde{X}]) &= \phi_*((\alpha \times [\tilde{X}]) \cap \phi^* \tilde{\omega}^*([\tilde{X}])) \\ &= \phi_*((\alpha \times [\tilde{X}]) \cap p_X^*([\tilde{X}])) \\ &= \phi_*(\alpha \times \{x\}) \end{aligned}$$

For this, we used the fact from (13) that $\tilde{\omega} \circ \phi$ is proper homotopic to $p_{\tilde{X}} : Z \times \tilde{X} \rightarrow \tilde{X}$. Thus our map is just $H_*(\tilde{A}) \xrightarrow{\rho_*} H_*(Z) \xrightarrow{(z \mapsto (z, x))_*} H_*(Z \times \tilde{X}) \xrightarrow{\phi_*} H_*(\tilde{A})$, which is clearly the identity map. \square

Theorem 16. *Notation and assumptions as in (9). Assume further that $A \rightarrow X$ is a generically finite complex analytic map of compact complex analytic spaces and that X is normal.*

Suppose that \tilde{A} is homotopy equivalent to a finite CW complex. Then $A \rightarrow X$ is a bundle with finite fiber.

Proof. Denote by Z the finite CW complex homotopy equivalent to \tilde{A} . Then Z has finitely many connected components, corresponding to the connected components of \tilde{A} . We have a representation $\pi_1(X) \rightarrow \text{Sym}(\pi_0(\tilde{A})) = \text{Sym}(\pi_0(Z))$. It suffices to prove the theorem in the case where we replace X with the finite covering corresponding to the kernel of this representation. In this case, the connected components of A correspond exactly to the connected components of \tilde{A} , and it suffices to prove the theorem for each of these separately. Thus we may assume without loss of generality that A and \tilde{A} are connected. Thus Z is connected as well.

By (15), we know that the homology of any fiber of $\tilde{A} \rightarrow \tilde{X}$ surjects onto $H_*(\tilde{A}) = H_*(Z)$. The former can be just a finite number of points, so we conclude that $H_i(Z) = 0$ for $i > 0$. Since Z is connected, we have $H_0(Z) = \mathbb{Z}$.

Fix a CW structure on \tilde{X} . On the chain level (for cellular cohomology), we have $C_c^*(Z \times \tilde{X}) = C^*(Z) \otimes C_c^*(\tilde{X})$. By the universal coefficient theorem, we have $H^0(Z) = \mathbb{Z}$ and $H^i(Z) = 0$ for $i > 0$. Thus the Künneth formula (see [Osb00, Thm.9.16]) implies there is an isomorphism $H_c^*(\tilde{X}) \rightarrow H_c^*(Z \times \tilde{X}) = H_c^*(\tilde{A})$. Thus we conclude that $\dim \tilde{A} = \dim \tilde{X}$ and that \tilde{A} is irreducible (\tilde{X} is irreducible since it is normal).

Now let us show that Z is simply connected. Suppose $(\mathbb{S}^1, \mathbf{0}) \rightarrow (Z, z)$ is any map of pointed spaces representing $\alpha \in \pi_1(Z, z)$. Then we get a corresponding map $f : \mathbb{S}^1 \times \tilde{X} \rightarrow Z \times \tilde{X} \rightarrow \tilde{A}$ (the latter map being the proper homotopy equivalence from (13)). Pick a point $a \in \tilde{A}$ where \tilde{A} is a manifold, and smooth f in a neighborhood of $f^{-1}(\{a\})$, so that (for a generic smoothing) $f^{-1}(\{a\})$ is a disjoint union of some number of circles, say C_1, \dots, C_k , each of which is transverse to $\mathbf{0} \times \tilde{X}$. Let $[\tilde{A}]$ denote the fundamental class of \tilde{A} , so that the natural map on H_*^{lf} (induced from $\tilde{X} \rightarrow Z \times \tilde{X} \rightarrow \tilde{A}$) sends $[\tilde{X}]$ to $[\tilde{A}]$. Now we may write:

$$1 = \langle \{a\}, [\tilde{A}] \rangle = \langle \{a\}, f(\mathbf{0} \times \tilde{X}) \rangle = \langle f^{-1}(\{a\}), \mathbf{0} \times \tilde{X} \rangle = \sum_i \langle C_i, \mathbf{0} \times \tilde{X} \rangle$$

Each term $\langle C_i, \mathbf{0} \times \tilde{X} \rangle$ is just the winding number of C_i around \mathbb{S}^1 . Equip each C_i which intersects $\mathbf{0} \times \tilde{X}$ with a base point on this intersection. Then each such C_i represents the element $\alpha^{(C_i, \mathbf{0} \times \tilde{X})}$ in $\pi_1(Z, z)$. On the other hand, C_i is mapped to $\{a\}$ in \tilde{A} , and thus is trivial in $\pi_1(\tilde{A}) = \pi_1(Z)$. Thus $\alpha^{(C_i, \mathbf{0} \times \tilde{X})}$ is trivial in $\pi_1(Z, z)$. Taking the product over all C_i with nonzero $\langle C_i, \mathbf{0} \times \tilde{X} \rangle$ and observing that the sum of these values is 1, we conclude that $\alpha \in \pi_1(Z, z)$ is trivial. Thus Z is simply connected.

Since Z is simply connected, and $H_i(Z) = 0$ for $i > 0$, we conclude that Z is contractible. Thus $\tilde{A} \rightarrow \tilde{X}$ is a homotopy equivalence. From this, we see that the map $A \rightarrow X$ induces an isomorphism on π_* , and thus is a homotopy equivalence as well. Now this implies that $\varpi_*([A]) = [X]$, so in particular the general fiber of $A \rightarrow X$ is a single point.

If A is projective and $\varpi^{-1}(\{x\})$ is positive dimensional, then $[\varpi^{-1}(\{x\})]$ is a nonzero class in $H_*(A)$ that gets killed in $H_*(X)$. This is a contradiction, hence $A \rightarrow X$ is an isomorphism.

In the general case, we argue as follows. Let $S \subset X$ be the set of $x \in X$ whose fiber $\varpi^{-1}(\{x\})$ is not a single point. Then S is a closed subvariety of X . Suppose that S is nonempty, and consider the natural morphism of long exact sequences:

$$\begin{array}{ccccccc} H_{*+1}(A) & \rightarrow & H_{*+1}(A, \varpi^{-1}(S)) & \rightarrow & H_*(\varpi^{-1}(S)) & \rightarrow & H_*(A) \rightarrow H_*(A, \varpi^{-1}(S)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{*+1}(X) & \rightarrow & H_{*+1}(X, S) & \rightarrow & H_*(S) & \rightarrow & H_*(X) \rightarrow H_*(X, S) \end{array}$$

A closed subvariety of a complex analytic space has an open neighborhood which deformation retracts onto it. Hence we can replace the relative homologies with the (reduced) homologies of the quotient spaces:

$$\begin{array}{ccccccc} H_{*+1}(A) & \rightarrow & \tilde{H}_{*+1}(A/\varpi^{-1}(S)) & \rightarrow & H_*(\varpi^{-1}(S)) & \rightarrow & H_*(A) \rightarrow \tilde{H}_*(A/\varpi^{-1}(S)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{*+1}(X) & \rightarrow & \tilde{H}_{*+1}(X/S) & \rightarrow & H_*(S) & \rightarrow & H_*(X) \rightarrow \tilde{H}_*(X/S) \end{array}$$

The map $A \rightarrow X$ is a homotopy equivalence, so the induced maps on H_* are isomorphisms. The map $A/\varpi^{-1}(S) \rightarrow X/S$ is a homeomorphism, so the induced maps on H_* are isomorphisms. Thus by the five lemma, the map $H_*(\varpi^{-1}(S)) \rightarrow H_*(S)$ is an isomorphism. However, X is normal so Zariski's main theorem implies that $\dim \varpi^{-1}(S) > \dim S$, contradicting the isomorphism $H_*(\varpi^{-1}(S)) \xrightarrow{\sim} H_*(S)$. Hence S is empty, that is $A \rightarrow X$ is an isomorphism of complex spaces. \square

2. SEMIALGEBRAIC COVERS

Definition 17 (Semialgebraic sets). See [BCR98, Chap.2] for a detailed treatment and for the results that we use.

A *basic open semialgebraic subset* of \mathbb{R}^n is defined by finitely many polynomial inequalities $g_i(x_1, \dots, x_n) > 0$. Using finite intersections and complements we get all semialgebraic subsets.

Let Y be a complex, affine, algebraic variety. Choose any (not necessarily closed) embedding $Y \subset \mathbb{C}^N$. Identifying \mathbb{C}^N with \mathbb{R}^{2N} we get the notion of semialgebraic subsets of \mathbb{C}^N and of Y . The latter is independent of the embedding. Thus we can talk about *semialgebraic subsets* of any complex algebraic variety. If $f : X \rightarrow Y$ is a morphism of varieties and $W \subset Y$ is semialgebraic, then so is $f^{-1}(W)$. By the Tarski–Seidenberg theorem, if $V \subset X$ is semialgebraic, then so is $f(V)$.

The closure, interior or a connected component of a semialgebraic set is again semialgebraic. Every compact semialgebraic set has a semialgebraic triangulation. Thus every open semialgebraic subset is homeomorphic to the interior of a finite polyhedron. In particular, it is homotopic to a finite CW complex.

An open semialgebraic subset of an algebraic variety Y is also naturally a complex space. We say that a complex space W is *semialgebraic* if it is biholomorphic to an open semialgebraic subset of a projective variety Y . Note that usually Y and the embedding are very far from being unique in any sense.

Remark 18. Gabrielov pointed out that the above properties are also shared by sets definable in an o-minimal theory; see [vdD98]. Thus our results have natural analogs in any o-minimal theory. It would be interesting to find some examples of universal covers that are o-minimal but not semialgebraic.

Definition 19 (Chow varieties). See [HP52, Secs.X.6–8] or [Kol96, Sec.I.3] for precise definitions and proofs.

Let Z be a projective variety over \mathbb{C} . An effective r -cycle is a formal linear combination $W = \sum_i m_i W_i$ where the W_i are irreducible r -dimensional subvarieties of Z and the m_i are natural numbers. The homology class of W is defined as $[W] := \sum m_i [W_i] \in H_{2r}(Z, \mathbb{Z})$.

For a homology class $\alpha \in H_{2r}(Z, \mathbb{Z})$, let $\text{Chow}_\alpha(Z)$ denote the Chow variety parametrizing those effective r -cycles whose homology class equals α . For an r -cycle W , the corresponding Chow point is denoted by $Ch(W) \in \text{Chow}_\alpha(Z)$. Then $\text{Chow}_\alpha(Z)$ is a projective algebraic variety and there is a universal family

$$\begin{array}{ccc} \text{Univ}_\alpha(Z) & \xrightarrow{\pi} & Z \\ u \downarrow & & \\ \text{Chow}_\alpha(Z) & & \end{array}$$

Fix next a normal, projective variety F and consider the subset $\text{Chow}_{(F, \alpha)}(Z) \subset \text{Chow}_\alpha(Z)$ parametrizing the images of embeddings $\tau : F \hookrightarrow Z$ such that $[\tau(F)] =$

α . Unfortunately, $\text{Chow}_{(F,\alpha)}(Z)$ need not be algebraic if F has an infinite discrete group of automorphisms. We can, however, easily remedy this problem. Fix ample divisors L (resp. H) on Z (resp. on F) and a number $C > 0$. We can then look at the images of embeddings $\tau : F \hookrightarrow Z$ such that $[\tau(F)] = \alpha$ and the intersection numbers $(L^i \cdot \tau^* H^j)_F$ are $\leq C$ for $i + j = \dim F$. (Note that this is essentially equivalent to bounding the degree, and thus the homology class, of the graph of τ in $F \times Z$ under the product polarization given by H and L .) These form a constructible algebraic subset $\text{Chow}_{(F,H,C,\alpha)}^\circ(Z, L) \subset \text{Chow}_\alpha(Z)$.

In order to avoid working with constructible sets, let $\text{Chow}_{(F,H,C,\alpha)}(Z, L)$ denote the closure of $\text{Chow}_{(F,H,C,\alpha)}^\circ(Z, L)$ in $\text{Chow}_\alpha(Z)$. There is a universal family

$$\begin{array}{ccc} \text{Univ}_{(F,H,C,\alpha)}(Z, L) & \xrightarrow{\pi} & Z \\ u \downarrow & & \\ \text{Chow}_{(F,H,C,\alpha)}(Z, L) & & \end{array}$$

where u is a fiber bundle with fiber F over $\text{Chow}_{(F,H,C,\alpha)}^\circ(Z, L)$.

The aim of this section is to prove the following.

Theorem 20. *Let Y be a smooth projective variety with universal cover \tilde{Y} . Assume that \tilde{Y} is contractible. Let $g : X \rightarrow Y$ be a morphism from a normal, projective variety X to Y and $\tilde{X} := \tilde{Y} \times_Y X$ the corresponding $\pi_1(Y)$ -cover. If \tilde{X} is open, semialgebraic in a projective variety \bar{X} then $g : X \rightarrow Y$ is a fiber bundle.*

We follow the arguments in (6) but there are some complications.

First, the map \tilde{g} need not extend to \bar{X} ; see (24). We thus pass from \bar{X} to $\text{Chow}_\alpha(\bar{X})$ where $\alpha := [\tilde{F}]$ is the homology class of a general fiber of \tilde{g} .

Second, there are no sensible moduli spaces for higher dimensional varieties in general, but $\text{Chow}_{(F,H,C,\alpha)}(\bar{X}, \bar{L})$ acts as a fiber of the (nonexistent) moduli map. Applying (21) to (roughly) the intersection $\tilde{X} \cap \pi(\text{Univ}_{(F,H,C,\alpha)}(\bar{X}, \bar{L}))$ we conclude that $\pi : \text{Univ}_{(F,H,C,\alpha)}(\bar{X}, \bar{L}) \rightarrow \bar{X}$ is onto, hence the fibers of $g : X \rightarrow Y$ are isomorphic to each other, at least over a Zariski open set.

Finally, in order to deal with the singular fibers of g , we prove that one can factor $\tilde{g} : \tilde{X} \rightarrow \tilde{Y}' \rightarrow \tilde{Y}$ where $\tilde{X} \rightarrow \tilde{Y}'$ is a fiber bundle and $\tilde{Y}' \rightarrow \tilde{Y}$ is generically finite. Then we apply (15) to $\tilde{Y}'/\Gamma \rightarrow Y$ to conclude that $\tilde{Y}' \rightarrow \tilde{Y}$ is in fact an isomorphism.

The next simple consequence of (14) will be very useful.

Lemma 21. *Notation and assumptions as in (20). Fix a normal, projective variety \bar{X} that contains \tilde{X} as an open semialgebraic subset. Let $\tilde{Z} \subset \tilde{X}$ be a nonempty closed, Γ -invariant, analytic subset that is semialgebraic in \bar{X} .*

Then $g : \tilde{Z}/\Gamma \rightarrow Y$ is surjective.

Proof. Note that \tilde{Z}/Γ is a closed analytic subset of the projective variety X , hence algebraic by Chow's theorem. Thus (14) applies to $g : \tilde{Z}/\Gamma \rightarrow Y$, hence $g : \tilde{Z}/\Gamma \rightarrow Y$ is surjective. \square

22 (Proof of (20)). The connected components of \tilde{X} are in one-to-one correspondence with the cosets of $\text{im}[\pi_1(X) \rightarrow \pi_1(Y)]$ in $\pi_1(Y)$. Since \tilde{X} is semialgebraic, there are only finitely many cosets. Thus, by passing to a finite cover of Y , we may assume that $\pi_1(X) \twoheadrightarrow \pi_1(Y)$ is surjective and hence \tilde{X} is connected. Note that g is surjective by (14).

Let $F \subset X$ be an irreducible component of a general fiber of g and let $\alpha := [F] \in H_*(X, \mathbb{Z})$ be the homology class of F . Fix $C \gg 1$ and let H be an ample line bundle on X ; we use H to denote $H|_F$ as well. Let $\text{Chow}_\alpha^*(X)$ (resp. $\text{Chow}_{(F,\alpha)}^*(X)$) denote the unique irreducible component of $\text{Chow}_\alpha(X)$ (resp. $\text{Chow}_{(F,H,C,\alpha)}(X, H)$) that contains the Chow point $Ch(F)$. (The notation $\text{Chow}_{(F,\alpha)}^*(X)$ indicates that this is independent of H and C .) By (19), $\text{Chow}_\alpha^*(X)$ and $\text{Chow}_{(F,\alpha)}^*(X)$ are projective varieties and there are universal families

$$\begin{array}{ccc} \text{Univ}_\alpha^*(X) & \xrightarrow{\pi} & X \\ u \downarrow & & \downarrow g \\ \text{Chow}_\alpha^*(X) & \rightarrow & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} \text{Univ}_{(F,\alpha)}^*(X) & \xrightarrow{\pi} & X \\ u \downarrow & & \downarrow g \\ \text{Chow}_{(F,\alpha)}^*(X) & \rightarrow & Y. \end{array} \quad (22.1)$$

Note that $\pi : \text{Univ}_\alpha^*(X) \rightarrow X$ is birational and an isomorphism over $y \in Y$ if $g^{-1}(y)$ is reduced and of dimension $= \dim F$. On the other hand, $\pi : \text{Univ}_{(F,\alpha)}^*(X) \rightarrow X$ is birational \Leftrightarrow it is surjective \Leftrightarrow there is a Zariski dense open set $Y^0 \subset Y$ such that the irreducible components of the fibers over Y^0 are all isomorphic to F .

We could complete the diagrams with Y in the lower right corner since every connected, effective cycle that is homologous to F is contained in a fiber of g .

Next we pull everything back to \tilde{Y} to obtain diagrams where all objects are analytic spaces and all morphisms are proper.

$$\begin{array}{ccc} U_\alpha(\tilde{X}) & \xrightarrow{\tilde{\pi}} & \tilde{X} \\ \tilde{u} \downarrow & & \downarrow \tilde{g} \\ C_\alpha(\tilde{X}) & \rightarrow & \tilde{Y} \end{array} \quad \text{and} \quad \begin{array}{ccc} U_{(F,\alpha)}(\tilde{X}) & \xrightarrow{\tilde{\pi}} & \tilde{X} \\ \tilde{u} \downarrow & & \downarrow \tilde{g} \\ C_{(F,\alpha)}(\tilde{X}) & \rightarrow & \tilde{Y}. \end{array} \quad (22.2)$$

Fix a normal, projective variety \bar{X} that contains \tilde{X} as an open semialgebraic subset and a lifting \tilde{F} of F . Note that \tilde{F} is an irreducible component of a general fiber of \tilde{g} . Let $\bar{\alpha} := [\tilde{F}] \in H_*(\bar{X}, \mathbb{Z})$ be the homology class of \tilde{F} .

Let $\text{Chow}_{\bar{\alpha}}^*(\bar{X})$ denote the irreducible component of $\text{Chow}_{\bar{\alpha}}(\bar{X})$ that contains the Chow point $Ch(\tilde{F})$.

Let \bar{L} be an ample line bundle on \bar{X} . Choose C large enough and let $\text{Chow}_{(F,\bar{\alpha})}^*(\bar{X})$ be the union of *all* irreducible components of $\text{Chow}_{(F,H,C,\bar{\alpha})}(\bar{X}, \bar{L})$ that contain the Chow point of any preimage of F . Since the irreducible components \tilde{F}'_y of general fibers of \tilde{g} are deformations of each other, the intersection numbers $(\bar{L}^i \cdot \tilde{H}^j \cdot \tilde{F}'_y)$ are independent of \tilde{F}'_y for $i + j = \dim F$. Thus if the Chow point of *one* preimage of F is in $\text{Chow}_{(F,H,C,\bar{\alpha})}(\bar{X}, \bar{L})$ then so is the Chow point of *every* preimage of F .

Thus $\text{Chow}_{(F,\bar{\alpha})}^*(\bar{X})$ is an algebraic variety, independent of C , and there are universal families

$$\begin{array}{ccc} \text{Univ}_{\bar{\alpha}}^*(\bar{X}) & \xrightarrow{\tilde{\pi}} & \bar{X} \\ \bar{u} \downarrow & & \\ \text{Chow}_{\bar{\alpha}}^*(\bar{X}) & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \text{Univ}_{(F,\bar{\alpha})}^*(\bar{X}) & \xrightarrow{\tilde{\pi}} & \bar{X} \\ \bar{u} \downarrow & & \\ \text{Chow}_{(F,\bar{\alpha})}^*(\bar{X}) & & \end{array} \quad (22.3)$$

(Note, however, that $\text{Univ}_{\bar{\alpha}}^*(\bar{X}) \rightarrow \bar{X}$ need not be birational and usually one can not complete the diagrams with any \bar{Y} in the lower right corner; see Example 24.)

The key step of the proof is the following.

Claim 22.4. Notation and assumptions as above. There are natural inclusions

$$\begin{array}{ccc} U_\alpha(\tilde{X}) & \hookrightarrow & \text{Univ}_{\bar{\alpha}}^*(\bar{X}) \\ \tilde{u} \downarrow & & \bar{u} \downarrow \\ C_\alpha(\tilde{X}) & \hookrightarrow & \text{Chow}_{\bar{\alpha}}^*(\bar{X}) \end{array} \quad \text{and} \quad \begin{array}{ccc} U_{(F,\alpha)}(\tilde{X}) & \hookrightarrow & \text{Univ}_{(F,\bar{\alpha})}^*(\bar{X}) \\ \tilde{u} \downarrow & & \bar{u} \downarrow \\ C_{(F,\alpha)}(\tilde{X}) & \hookrightarrow & \text{Chow}_{(F,\bar{\alpha})}^*(\bar{X}). \end{array}$$

Under these inclusions, the images of $U_\alpha(\tilde{X})$, $C_\alpha(\tilde{X})$, $U_{(F,\alpha)}(\tilde{X})$, $C_{(F,\alpha)}(\tilde{X})$ are open semialgebraic subsets.

Proof. It is enough to prove that the image of $U_\alpha(\tilde{X})$ is an open semialgebraic subset of $\text{Univ}_\alpha^*(\tilde{X})$.

As we noted in (17), $\pi^{-1}(\bar{X} \setminus \tilde{X}) \subset \text{Univ}_\alpha^*(\bar{X})$ and the complement of its projection

$$W_\alpha := \text{Chow}_\alpha^*(\bar{X}) \setminus u(\pi^{-1}(\bar{X} \setminus \tilde{X})) \subset \text{Chow}_\alpha^*(\bar{X})$$

are semialgebraic. Note that W_α parametrizes those cycles $Z \subset \bar{X}$ that are contained in \tilde{X} and satisfy $[Z] = \bar{\alpha}$. Such a Z is in $U_\alpha(\tilde{X})$ iff its image in Y is 0-dimensional. The latter condition is invariant under deformations, hence $U_\alpha(\tilde{X})$ is a connected component of W_α , hence semialgebraic. \square

We can thus apply (21) to $C_{(F,\alpha)}(\tilde{X}) \rightarrow \tilde{Y}$ to conclude that the composite $C_{(F,\alpha)}(\tilde{X}) \rightarrow \tilde{Y} \rightarrow Y$ is surjective and so is $\text{Chow}_{(F,\alpha)}^*(X) \rightarrow Y$. Thus there is a Zariski dense open set $Y^0 \subset Y$ such that the irreducible components of the fibers over Y^0 are all isomorphic to F .

Let $Z \subset \text{Chow}_{(F,\bar{\alpha})}^*(\bar{X})$ be the Zariski closure of the complement of $\text{Chow}_{(F,\bar{\alpha})}^o(\bar{X})$. Then $Z \cap \tilde{X}$ is a closed analytic subset, invariant under Γ , and its image in Y is contained in $Y \setminus Y^0$. Thus, by (21), $Z \cap \tilde{X}$ is empty. Therefore $U_\alpha(\tilde{X}) \rightarrow C_\alpha(\tilde{X})$ and $u : \text{Univ}_\alpha(X) \rightarrow \text{Chow}_\alpha(X)$ are fiber bundles with fiber F .

The set of points where $\text{Univ}_{(F,\bar{\alpha})}^*(\bar{X}) \rightarrow \bar{X}$ is not étale is closed, its intersection with \tilde{X} is a closed analytic subset, invariant under Γ and its image in Y is contained in $Y \setminus Y^0$. Using (21) again we see that $U_\alpha(\tilde{X}) \rightarrow \tilde{X}$ and $\text{Univ}_\alpha(X) \rightarrow X$ are isomorphisms. Thus \tilde{g} factors as

$$\tilde{g} : \tilde{X} = U_\alpha(\tilde{X}) \xrightarrow{\tilde{u}} C_\alpha(\tilde{X}) \rightarrow \tilde{Y}.$$

Here $C_\alpha(\tilde{X})$ is semialgebraic and $C_\alpha(\tilde{X})/\Gamma = \text{Chow}_\alpha(X)$ has a generically finite morphism to Y . By (15) we conclude that $\text{Chow}_\alpha(X) \rightarrow Y$ is an isomorphism.

Thus $g : X \rightarrow Y$ is isomorphic to the fiber bundle $u : \text{Univ}_\alpha(X) \rightarrow \text{Chow}_\alpha(X)$. \square

23 (Proof of Theorem 3).

If (3.2) holds then \tilde{X}_Γ is a holomorphic fiber bundle over \mathbb{D} with fiber F . By [Gra58], every fiber bundle over a contractible Stein space is trivial, giving (3.3).

Every bounded symmetric domain is semialgebraic, hence (3.3) implies (3.1).

Finally, assume (3.1). By Lemma 8 we have a holomorphic map $X \rightarrow \mathbb{D}/\Gamma$ and it is a fiber bundle by Theorem 20. \square

Example 24. Let $\mathbb{B}^2 \subset \mathbb{C}^2$ denote the unit ball. Then $\mathbb{P}^1 \times \mathbb{B}^2$ is the universal cover of an algebraic threefold. It can also be realized as an open semialgebraic subset of a smooth cubic 3-fold $\bar{X}_3 \subset \mathbb{P}^4$ such that the fibers of $\mathbb{P}^1 \times \mathbb{B}^2 \rightarrow \mathbb{B}^2$ become lines in \bar{X}_3 . The family of all lines on \bar{X}_3 is irreducible and there are 6 lines through a general point. Thus the projection $\mathbb{P}^1 \times \mathbb{B}^2 \rightarrow \mathbb{B}^2$ will not extend to \bar{X}_3 in any way.

3. OPEN PROBLEMS

Here we discuss a possible approach to Conjecture 2 and several other questions that, directly or indirectly, relate to it. We also discuss which arguments in [CHK11] generalize and which need further new ideas.

The first part of the proof in [CHK11] shows that if \tilde{X} is quasi-projective then $\pi_1(X)$ has a finite index abelian subgroup. One can follow the proof given in [CHK11] until the point where we need to exclude the case when X is of general type. If \tilde{X} is quasi-projective, this is done using Kobayashi-Ochiai [KO75].

In the semialgebraic case there is no contradiction. In fact, if \mathbb{U} is a bounded open subset of a Stein manifold and Γ is a group acting properly discontinuously and freely with compact quotient, then \mathbb{U}/Γ is a projective variety with ample canonical class. (This is essentially due to Poincaré; see [Kol95, Chap.5] for details.) While this sounds very general, there are few such examples known aside from bounded symmetric domains [Šab77]. The results of [Vey70, Won77, Fra89] show that under various additional restrictions, such a \mathbb{U} is necessarily a bounded symmetric domain; see [IK99] for a survey of closely related results. From our point of view, this leads to the following problem.

Question 25. Let \mathbb{U} be a semialgebraic bounded open subset of an affine variety. Assume that there is a group Γ acting properly discontinuously with compact quotient. Is \mathbb{U} necessarily a bounded symmetric domain?

In general one can hope that (25), together with the conjectures [Kol95, 18.6–8], imply that if X is a smooth projective variety whose universal cover is biholomorphic to a semialgebraic subset of a projective variety then $\pi_1(X)$ is commensurate with an extension of an abelian group with a cocompact lattice acting on a bounded symmetric domain.

The second part of the proof in [CHK11] deals with the case when $\pi_1(X)$ is a free abelian group \mathbb{Z}^{2r} . Then the Albanese map $\text{alb}_X : X \rightarrow \text{Alb}(X) \cong \mathbb{C}^r/\mathbb{Z}^{2r}$ is the natural candidate for the fiber bundle structure in (3.2). In our case, we construct the non-abelian Albanese map by referring to the works of [ES64] and [Siu82]. We thank D. Toledo for explaining to us that the case when \mathbb{D} is a reducible, bounded, symmetric domain of dimension ≥ 2 and Γ is a cocompact, irreducible lattice acting on \mathbb{D} can be handled using the results of [Mok85].

Thus we obtain an algebraic morphism $g : X \rightarrow \mathbb{D}/\Gamma$. The relative version of the Albanese morphism should deal with the (almost abelian) kernel of $\pi_1(X) \rightarrow \Gamma$.

The third step is to prove that the morphism $g : X \rightarrow \mathbb{D}/\Gamma$ is a fiber bundle. The proof in [CHK11] relied on a detailed knowledge of subvarieties of Abelian varieties and their finite ramified covers. This is replaced by the topological arguments of Section 1.

While Theorems 15 and 20 are more general than needed for our purposes, all the examples suggest that even stronger results may be true.

Question 26. Let Y be a smooth projective variety and $g : Y \rightarrow X$ a (not necessarily surjective) morphism to a (not necessarily smooth) projective variety. Assume that the universal cover \tilde{X} is contractible and that the fiber product $\tilde{Y} := Y \times_X \tilde{X}$ is homotopic to a finite CW complex.

Is then Y a differentiable fiber bundle over X ?

The example of Kodaira fibrations (see, for instance, [BPVdV84, Sec.V.14]) shows that $g : Y \rightarrow X$ need not be a holomorphic fiber bundle.

The above question leads naturally to several interesting problems concerning the topology of algebraic maps; these are discussed in [FdBK12].

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